Transition among synchronized states mediated by attractor-repeller collision crisis

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We studied the transition from a single-value generalized synchronization state to a double-value one in a unidirectionally coupled two-dimensional map system. It is found that this discontinuous transition is mediated by the attractor-repeller collision crisis and is different from the blowout bifurcation in many respects. By using the unstable periodic orbits decomposition method, it is shown that the attractor is generally nondifferential in the parameter regime about the transition. Based on the nondifferential character of the attractor, we propose a mechanism for the attractor-repeller collision crisis.

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Synchronization of mutually coupled units is found abundant in natural situations such as flashing of fire flies [5], pacemaker cells of the heart [6], etc., and laboratory studies such as laser dynamics [7], electric circuits [8], chemical reactions [1], and secret communication [2]. In the past decades study on chaotic synchronization [3,4] attracted a lot of research interest and became one of the hottest subfields of nonlinear dynamics.

In the literature, the transition from a synchronized state to a desynchronized one, named blowout bifurcation [9], is already well studied. It is found that, about this symmetry-breaking bifurcation [10], interesting phenomenon including on-off intermittency [11], riddled basin of attraction [12], and unstable dimension variability [13] appear. On the other hand, another transition frequently met in synchronization study is nearly ignored, i.e., the transition between synchronized states [14,15]. For this case, the synchronous manifold is transversely stable on both sides of the transition. Therefore, it should be totally different from the synchronization-desynchronization transition, which takes place when the synchronous manifold loses its transverse stability.

In this paper, we show a transition from a single-value generalized synchronization state to a double-value one [16] in a unidirectionally coupled map system. By using the unstable periodic orbit decomposition method [17], it is shown that the attractors are nondifferential about the transition. Owing to the nondifferential character, a different scenario of the attractor-repeller collision crisis, from the one reported in [18,19], is proposed. We argue that this scenario is generic for transitions among chaotic states.

Our system is the parametrically driven logistic map

$$x_{n+1}=\mu f(x_n),$$
$$y_{n+1}=z(x_n)f(y_n), \qquad (1)$$

where $f(x)=x(1-x)$, $z(x)=(ax+b)$, $\mu=3.9999$, a, and b are positive real constants. In the zero-coupling limit $a\to 0$, the y subsystem is decoupled from the chaotic driving x_n and it exhibits a cascade of period-doubling bifurcations on increasing parameter b. It is expected that with weak coupling, there should be a certain transition corresponding to the period-doubling bifurcation. For the simplicity of illustration, here we focus our attention on the parameter regime about the first period-doubling bifurcation. The same mechanism should work at all the other period-doubling bifurcations. In a former study on a randomly driven system we found that the period-doubling bifurcation is replaced by a crisis [20] named tunnel crisis [14]. For the current chaotically driven case, the complex internal structure of the driving signal, i.e., the large number of unstable periodic orbits embedded in the chaotic attractor [17], makes the dynamics richer and interesting.

We first calculate the bifurcation diagram of y_n with $a=0.3$ (Fig. 1). On increasing the parameter b, a one-band attractor bifurcates to a two-band one at a certain value of b. As a standard method used in the study of generalized synchronization, we study an ensemble of identical y subsystems driven by the same chaotic signal x_n. Below the bifurcation, an ensemble of identical y subsystems starting from different initial conditions [21] collapse to a single trajectory finally, although they behave chaotically with time going on. This is the single-value generalized synchronization state. Beyond the bifurcation, the large number of identical y subsystems

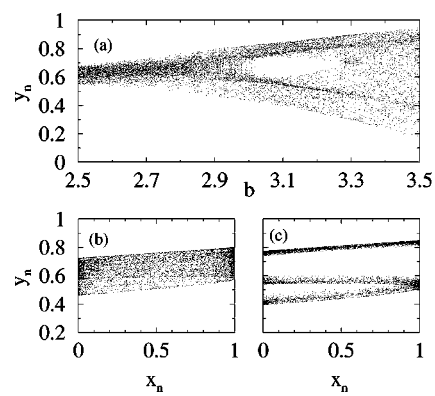

FIG. 1. (a) The bifurcation diagram of y_n with $a=0.3$; (b) the single-value synchronized state at $b=2.9$, and (c) the double-value synchronized state at $b=3.1$.

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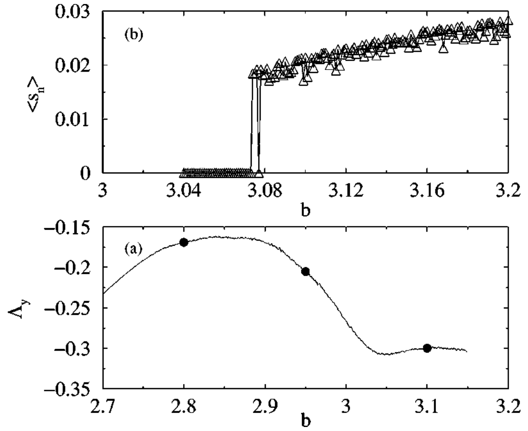

FIG. 2. (a) The subsystem Lyapunov exponent Λ_y and (b) the average value of s_n with increasing the parameter b.

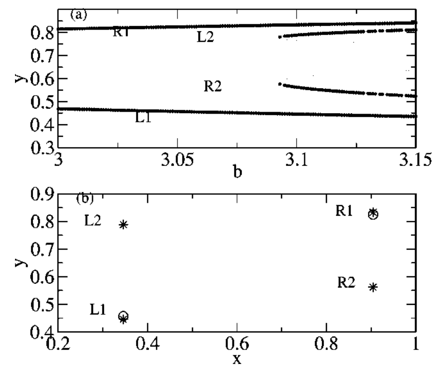

FIG. 3. (a) The bifurcation diagram for the period-2 UPO. The stable branches are in bold line while the unstable one is in thin line. (b) The position of stable branches for two cases with $b=3.05$ below the bifurcation (○) and with $b=3.1$ beyond the bifurcation (⋆).

split into two subgroups. The units in the same subgroup take an identical orbit while the two subgroups behave independently. This is the double-value generalized synchronization state [16]. The subsystem Lyapunov exponent Λ_y calculated according to the equation

$$\Lambda_y=\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|z(x_n)df(y_n)/dy_n| \tag{2}$$

is shown in Fig. 2(a). One can see that Λ_y is negative on both sides of the transition and is far away from the zero-value line. It means that here the subsystem y_n is generalized synchronous to subsystem x_n on both sides of the bifurcation and the transition is directly from one synchronized state to another [22].

To characterize the transition quantitatively, here we define a quantity s_n that measures the average distance among an ensemble of trajectories $y_n^{(i)}$ starting from different initial conditions,

$$s_n=\sqrt{\frac{1}{L}\sum_{i=1}^{L}(y_n^{(i)}-\overline{y}_n)^2}, \tag{3}$$

where $\overline{y}_n=1/L\Sigma_{i=1}^{L}y_n^{(i)}$, i, and L are the index and total number of trajectories, respectively. s_n would be zero for the single-value synchronized state and nonzero for the double-value one. Thus, it can be used as an order parameter for the transition studied here. The average value of s_n on increasing the parameter b is shown in Fig. 2(b). At $b\simeq3.08$, it has a sudden jump from zero to a nonzero finite value, i.e., the transition encountered here is a discontinuous one. This is different from the supercritical blowout bifurcation where s_n increases linearly from zero as the system enters the desynchronized state [11]. We calculated also the basin of attraction of the attractor, it is not riddled as for the subcritical case of blowout bifurcation.

To unfold this complex transition of the chaotic set, we consider the unstable periodic orbits (UPOs) embedded in it [17]. As an example, first consider the period-2 UPO of the subsystem x_n. The bifurcation diagram for this UPO on increasing b is shown in Fig. 3. A bifurcation from the one-to-one response to the two-to-one response in the y subsystem occurs at $b\simeq3.096$. Due to the period-2 modulation of z_n in the y subsystem, the period-doubling bifurcation at zero coupling is rendered imperfect now. At the critical value $b\simeq3.096$, a pair of branches are created via a saddle-node bifurcation and the previously existing one is continuously stable through the bifurcation, and the new branches appearing at the bifurcation are at a finite distance from the existing one. This imperfect bifurcation is the right origin of the discontinuous transition encountered for the chaotic set. One should also note that the newly created pair of branches have a different number of unstable directions: one is a saddle having one unstable direction while the other is a repeller having two unstable directions.

At the limit of zero coupling, the subsystem y_n is decoupled from x_n, and all the UPOs on the chaotic set of x_n have their period-doubling bifurcation in the y direction at the same value of b. For nonzero coupling, the modulation in z_n of y subsystem has two effects: The first one is that it renders the period-doubling bifurcation imperfect as mentioned above. The second is that each UPO encounters the bifurcation at different values of the parameter b. Therefore, there is a bifurcation interval for the parameter b lasting from the bifurcation point of the first UPO to that of the last one. For b in this interval, the attractor of the system is of unstable dimensional variability (UDV) [13]. Another important feature of a system with parameter in this interval is that it is always single-value synchronized. If you cut the attractor at a certain place $x=x_0$, you get always one single point on the intersection. However, if the position of the intersection is shifted a tiny distance δ, the difference between the two intersection points $y(x_0)$ and $y(x_0+\delta)$ can be finite even as δ goes to zero. This is due to the fact that x_0 and $x_0+\delta$ may be on two UPOs which bifurcate in the y direction at different values of b. According to the definition given in Ref. [23], here the generalized synchronization is *nondifferential* in nature. Actually, nondifferential is not an exclusive feature

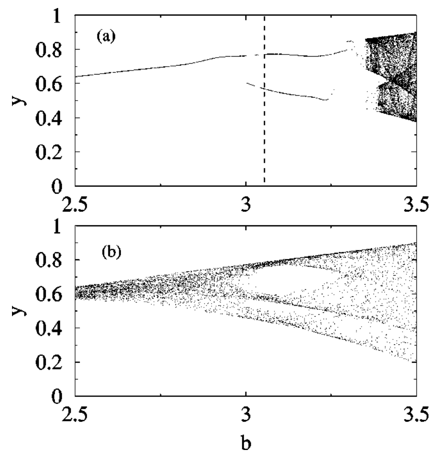

FIG. 4. The bifurcation diagram of y on the intersection $x_0 = 0.32$ and for $|x - x_0| < 10^{-4}$. The dashed line marks the transition from the single-value synchronized state to the double-value one.

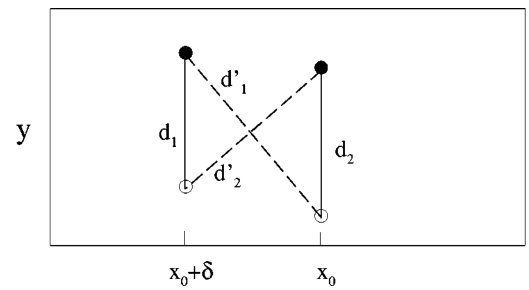

FIG. 5. The scheme for the definition of distances among unstable branch (○) and stable branch (●) of two UPOs at $x = x_0$ and $x = x_0 + \delta$.

of the single-value synchronized state in the bifurcation interval. Due to the same reason that UPOs bifurcate in the y direction at different times, the double-value synchronized state beyond the transition is also nondifferential. To illustrate this fact, we show in Fig. 4 the evolution of the intersection at $x_0 = 0.32$ with increasing b and the case for interaction $|x - x_0| < \delta$ with $\delta = 10^{-4}$. Owing to the nondifferential character of the attractor, the points recorded in the second case scatter greatly and the diagram cannot even be distinguished from the one shown in Fig. 1. The nondifferential feature can also be seen from the first case, where the intersection point has intermittent jumps with increasing b.

Now let us consider the relation between the bifurcation of the UPOs and the transition for the chaotic set. In the double-value synchronization regime, the system has two coexisting attractors. The boundary of their basins of attraction is a repeller set formed by unstable branches from the bifurcation of all UPOs. On decreasing the parameter b, this repeller set approaches the attractors gradually. At the moment when the distance between the attractor and the repeller set is zero, two attractors suddenly expand in size and merge into one. Following similar arguments given above about the attractor, one can get to know that the repeller set is also nondifferential in the x direction. The nondifferential property of the attractor and the repeller set makes the definition of the distance between them quite complex. One natural proposal is based on the UPO decomposition: For each UPO one can get a distance between its stable branch and the unstable branch. The minimal value of these distances for all UPOs is the distance between the chaotic attractor and the basin boundary (repeller) set. On decreasing the parameter b, stable branches and unstable ones approach gradually. At a certain value of b, one pair of them belonging to a certain UPO hit and annihilate each other. Then, a hole opens on the basin boundary and two attractors are no longer separated and merge into one. This is just the situation reported in [19,18]. The collision of the attractor and the boundary set is a local event, which happens at only the location of the pair of branches holding the minimal distance. This is the right reason why the characteristic time for this crisis is extremely long when comparing to the normal case of crisis [20].

However, this is not the only case encountered here, in general. The reason is as follows. Consider an intersection of the attractor at a certain place $x = x_0$ with x_0 on an UPO A. One can get a distance d_1 between the unstable branch and stable branch of this UPO. Now consider another intersection at $x = x_0 + \delta$ which is on a certain UPO B. One gets another distance d_2 similar to d_1 for A. In addition to these two distances, there are two more distances $d'_{1,2}$ which are important for our problem: the distance between unstable and stable branches from different UPOs (see Fig. 5). In the limit $\delta \to 0$, the four distances become identical if the attractor and the boundary set are both smooth. Then, one encounters the scenario of the transition mentioned above. For the case of nondifferential attractor and boundary set, these four are in general different. If the minimal one of them is from $d_{1,2}$, one encounters again the scenario given above. Otherwise, an unstable branch from a certain UPO will collide with a stable branch from the other UPO before any pair of branches from the same UPO hit each other together: As a result of this collision, two attractors suddenly merge into a big one. Here, due to the nondifferential nature of the attractor and basin boundary (repeller) set, an alternative scenario of the attractor-repeller collision crisis is expected. Since the only essential factor leading to the nondifferential is that the UPOs bifurcate at different parameter values, nondifferential character is expected to be a common feature for chaotic attractors about the transition. And consequently, the scenario proposed here is expected to be a generic one for the transition between synchronized states. Actually, our proposal for the distance among two fractal sets is quite general and is also applicable to strange nonchaotic sets.

Finally, we would like to outline the main results of this paper.

(1) We show an example of the transition between synchronized states.

(2) This transition is a discontinuous one and is different from the synchronization-desynchronization transition in many respects.

(3) By using the unstable periodic orbits (UPOs) decomposition method, we show that the synchronized state and their basin boundary are nondifferential in the parameter re-

gime about the transition. This nondifferential character of the attractor is expected to be generally found for the transitions between synchronized states.

(4) Based on the nondifferential nature of the attractor and repeller set, we propose a mechanism for the attractor-repeller collision crisis. It is expected to be a generic scenario for the transition among fractal (chaotic or strange nonchaotic) sets.

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[21] Throughout this paper, different initial condition means only the difference in y subsystem while the initial condition for the x subsystem is identical always.

[22] The transition from single-value synchronized state to a double-value one in Ref. [16 (b)] is the combination of two synchronization-desynchronization transitions.

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